Topics in Combinatorics MATH180B:  
Symmetric Groups, Symmetric functions and Schur Processes

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Mode of instruction: This will be an online synchronous class. Your online presence during class time (T, Th 4:00PM – 5:30 PM) is mandatory. The zoom link is indicated on Latte.

Equipment needs for successful remote participation: Classes and office hours will be held via Zoom. Assignment will be posted and returned on Latte. Student in this class are expected to have a computer with a camera, to be used during lectures and office hours. If you have trouble getting access to such material, please check with your instructor about the availability of emergency fund resources. Please also refer to the Online etiquette document on the Latte page.

Course description and objective: This class will explore the combinatorics of the symmetric groups, and two closely related topics: symmetric functions and probability distributions on integer partitions. The symmetric groups (the groups of permutations) are of great importance because they contain every finite group, and also because of their relations with groups of matrices. The representation theory of the symmetric group is a very rich subject. This theory can be approached using bijections and algorithms on permutations such as the Robinson-Schensted-Knuth correspondence. The symmetric functions also appear naturally in this theory, and again many interesting aspects of symmetric functions can be described in terms of bijections and algorithms. One of the bases of the vector space of symmetric functions in $n$ variables are the Schur functions. The Shur functions are indexed by integer partitions and can be used to define interesting probability measures, called Schur processes, on integer partitions and sequences of integer partitions. Schur processes have nice properties and have interpretations in terms of statistical mechanics model.

Learning goals: In this class we will (tentatively) cover:
- The different bases of symmetric functions.
- Some bits of the general theory of representations of finite groups.
- The irreducible representations of the symmetric group.
- The Robinson-Schensted-Knuth correspondence, and its generalizations.
- The combinatorics of Young tableaux and the connection coefficients in the symmetric group
- The Schur processes and their connections with physical models like the TASEP (totally asymmetric exclusion model).

See below for a detailed list of topics

**Requirement:** The class is open to every graduate student: graduate students in physics or computer science are welcome! The class is open to highly motivated undergraduate students (this is a hard-fast paced class) having already completed classes in proofs (MATH23), probability (MATH36A), and algebra (MATH30A or MATH28A+28B or an higher class).

**Reference:** We will not closely follow a particular book, but good references are Richard Stanley’s *Enumerative combinatorics Vol 2*, and Bruce Sagan’s *The Symmetric group*.

**Grading Policy:** The grade will be based on the weekly homework assignments.

**Office hours:** To be held via Zoom Fridays 1pm-2pm (Time to be confirmed) and by appointment.

**Expectation of students’ effort:** Success in this course is based on the expectation that students will spend a minimum of 2 hours of study time per for each hour of class (reviewing class material, completing homeworks, preparation for exams, etc.).

**Disabilities:** If you are a student with a documented disability on record at Brandeis University and wish to have a reasonable accommodation made for you in this class, please see me immediately.

**Academic Integrity:** You are expected to be familiar with, and to follow, the University’s policies on academic integrity. Please consult Brandeis University Rights and Responsibilities for all policies and procedures. All policies related to academic integrity apply to in-class and take home projects, assignments, exams, and quizzes. Students may only collaborate on assignments with permission from the instructor. Allegations of alleged academic dishonesty will be forwarded to the Director of Academic Integrity. Sanctions for academic dishonesty can include failing grades and/or suspension from the university.
Topics to be covered – tentative (optimistic) plan

1 Symmetric Group and RSK bijection (4 hours)

1. Basic definitions
   Def of symmetric groups and notations. Partitions. Conjugacy class = cycle type. Size of conjugacy class \( n!/z_\lambda \).

2. Tableaux and the Robinson-Schensted-Knuth correspondence
   Young diagrams, SYT, SSYT. The RSK bijection between \( S_n \) and set of pairs of SYT of same shape. Extension to bijection between \( n \times n \) matrix of integers (with fixed line/column sums) and pairs of SSYT of same shape (with given contents).

3. Hook-length formula (Green-Nijenhuis-Wilf proof)

2 Symmetric functions (6 hours)

1. Generating functions
   Power series. Combinatorial classes and generating functions. Union, Cartesian product, sequences of combinatorial classes in terms of generating functions. Generating function of partitions.

2. Algebra of symmetric functions
   Def of symmetric functions. Monomial basis \( m_\lambda \).

3. Elementary, Homogeneous and Power bases
   Def of \( e_\lambda, h_\lambda, p_\lambda \). Combinatorial interpretation of coef \( [m_\lambda]e_\mu \) and \( [m_\lambda]h_\mu \). Proof that \( e_\lambda \) basis. Generating functions \( E(t) = \sum_n e_nt^n \), \( H(t), P(t) \) and relations between \( e_n, h_n, p_n \). Duality \( \omega : e_n \mapsto h_n \) and proof that \( w(p_\lambda) = \text{sign}(\lambda)p_\lambda \).

4. Schur functions
   Definitions in terms of SSYT's. Proof that the Schur functions are symmetric and form a basis.

5. Cauchy formulas
   Proof of Cauchy formula based on RSK. Proof of Cauchy dual formula based on RSK dual (by column insertion) and that \( w(s_\lambda) = s_{\lambda'} \). Definition of the scalar product with dual bases \( \{h_\lambda\}/\{m_\lambda\}, \{p_\lambda\}/\{p_\lambda/z_\lambda\}, \) and \( \{s_\lambda\}/\{s_\lambda\} \).

6. Jacobi Trudi formula

3 Representations of finite groups - a review (5 hours)

1. Definitions and examples of representations
   Representation corresponding to a \( G \)-set. Regular representations. Equivalence of representations.

2. Irreducible representations
Any repress decomposes as sums of irreducible representations.

3. **Schur Lemma and consequences**
   Multiplicity of an irreducible representation \( U \) in a representation \( V \) is \( \dim(\text{Hom}_G(U,V)) \).

4. **Canonical isomorphism for the group algebra**

5. **Characters**
   Irreducible characters form an orthonormal basis of the vector space of class functions.
   Frobenius formula for connection coefficients.

6. **Restricted and induced representations**
   Definitions and Frobenius reciprocity formula.

### 4 Representations of \( S_n \) (5 hours)

1. **Specht modules**
   Definitions of the Specht as a subspace of \( C[S_n] \) via Young’s projectors.

2. **A basis for the Specht modules**

3. **Branching rule**
   Proof of the branching rule. Young lattice and Gelfand-Tsetlin basis.

### 5 Characters of \( S_n \) and Schur functions (6 hours)

1. **Frobenius characteristic map**
   Definition of the characteristic map as the linear map from the space \( CF_n \) of class function on \( S_n \) to the space of symmetric functions of deg \( n \) sending \( \pi \) to \( \frac{\pi^\lambda}{z_\lambda} \) where \( \lambda \) is the cycle type of \( \pi \). Proof that it sends characters to Schur functions.

2. **Classical definitions of Schur functions**
   Proof that the classical definition and combinatorial definition of Schur function coincide. Pieri’s rule.

3. **Murnaghan-Nakayama rule**

4. **Application to the enumeration of factorizations of the long cycle into transposition (Jackson formula)**

### 6 Knuth equivalence, Littlewood-Richardson coefficients and plane partitions (6 hours)

1. **Knuth equivalence and Greene theorem**

2. **Jeu de taquin and Littlewood-Richardson coefficients**

3. **Operator approach for RSK and Fomin’s growth diagrams**

4. **MacMahon formula for plane partitions**
7 Schur processes (7 hours)

1. Schur measures
Definition of Schur measure. Classification of Schur-positive evaluations. Examples of probabilistic models leading to Schur measures.

2. Schur processes

3. Studying Schur processes as determinantal point processes
Determinantal point processes and contour integrals. Application to the study of the poissonized Plancherel measure on partitions and determination of Kerov-Vershik limiting curve.