Math 180B: Topics in Combinatorics
Fall 2018.

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This year Math 180B will focus on two interwoven plots:

- Partitions, Young diagrams and tableaux.
- Combinatorial models from Mathematical Physics.

For both topics we will talk about related algebraic structures, enumeration, bijections, algorithms.

Lecture Notes.
There is no textbook for this course. I will write and post lecture notes.

Prerequisites.
You don’t need to know beforehand anything more advanced than determinants / eigenvalues. However, some previous familiarity with discrete mathematics (graphs, bijections, counting problems) will be helpful.

Grades.
Your grade in the course will be based on the following:

1. Homework (20% of your grade). One problem/exercise per week.
2. Short expository paper and presentation at the end of the term on a topic chosen together with the instructor (80% of your grade).

LATTE.
All course materials for Math 39a will be available online on LATTE. Log in at http://latte.brandeis.edu using your Unet username and password.

Four-Credit Course (with three hours of class time each week).
Success in this 4 credit hour course is based on the expectation that students will spend a minimum of 9 hours of study time per week in preparation for class (readings, papers, discussion sections, preparation for exams, etc).

Students with disabilities.
If you are a student who needs academic accommodations because of a documented disability you should contact me and present your letter of accommodation as soon as possible. If you have questions about documenting a disability or requesting academic accommodations you should contact Beth Rodgers-Kay in the Office of Academic Services at 63470 or at brodgers@brandeis.edu. Letters of accommodations should be presented at the start of the semester to ensure provision of accommodations. Accommodations cannot be granted retroactively.

Academic Integrity.
You are expected to follow the University’s policy on academic integrity, which is distributed annually as Section 4 of the Rights and Responsibilities Handbook (see http://www.brandeis.edu/studentaffairs/srcs/rr/index.html). Instances of alleged dishonesty will be forwarded to the Department of Student Development and Conduct for possible referral to the Student Judicial System. Potential sanctions include failure in the course and suspension from the University. If you have any questions about how these policies apply to your conduct in this course, please ask.

Anything interesting?.
Here is a brief overview of a cool story that I hope to tell you more about during this term.
Long story of longest increasing subsequences

• 1935. Paul Erdos and George Szekeres proved

Theorem: From any permutation of $1,2,\ldots,k^2+1$ it is always possible to choose either increasing or decreasing subsequence of length $k+1$.

Their paper begins "Our present problem has been suggested by Miss Esther Klein in connection to ...". Connected problem from combinatorial geometry became known as the "happy ending problem", because it led to the marriage of Szekeres and Klein in 1937.

• 1961. Stanislaw Ulam suggested studying distribution of the length $\ell_n$ of longest increasing subsequence of random permutation of $1,2,\ldots,n$. To approximate the average he performed numerical simulations and conjectured that $E[\ell_n]$ grows linearly in $\sqrt{n}$. This question was brought up as an example of the kinds of problems that can be attacked using Monte Carlo method (that Ulam pioneered). A quote: "An electronic computing machine like the IBM 704 will process several thousand of permutations of 101 integers in a few minutes."

• 1970. John M. Hammersley proved

Theorem: $\frac{\ell_n}{\sqrt{n}} \to C$ in probability, where $C$ is some constant, such that $\frac{2}{7} \leq C \leq e$.

The corresponding article is titled "A few seedlings of research". A quote: "So I have conceived the present paper as if it were one chapter for a companion volume, yet to be compiled, on the how of the mathematical research. ... This is how the seeds of research might be sown. Chance sows, and curiosity nurtures."

Hammersley suggests three non-rigorous ways to compute $C$: two give $C = 2$ and one gives $C = 2^{\frac{2}{3}}e^{\frac{1}{2}} = 1.961\ldots$.

• 1977. Anatoly Vershik and Sergei Kerov and independently Benjamin F. Logan and Lawrence A. Shepp proved

Theorem: $C = 2$.

Key idea: $\ell_n$ is distributed as length of the first row of a Young diagram $\lambda^n$ drawn according to the Plancherel measure, $P(\lambda^n = \lambda) = \frac{d_\lambda}{n!}$, where $d_\lambda$ is dimension of the irreducible representation of symmetric group $S_n$ of type $\lambda$. This can be proved via the Robinson-Schensted bijection.

Figure 3. Left: Young diagram $(4, 4, 3, 1, 1)$. Center: Young diagram sampled from Plancherel measure of order $n = 100$. Right: Young diagram sampled from Plancherel measure of order $n = 1000$. 

Theorem: \( \ell_n - 2\sqrt{n} \) is of order \( n^{1/6} \). Moreover, \( \frac{\ell_n - 2\sqrt{n}}{n^{1/6}} \to TW_2 \), where \( TW_2 \) is a Tracy-Widom probability distribution.

• 1999. Andrei Okounkov proved that the joint distribution of the first few rows of a random Young diagram drawn according to the Plancherel measure of order \( n \), is the same, after proper shift and rescaling, as the distribution of the first few eigenvalues of a gaussian random hermitian matrix of order \( n \). The proof involves comparing random surfaces given by Young diagrams and by ramified coverings and contains many ideas that anticipate Okounkov’s later work on Gromov-Witten invariants. In 2006 he was awarded the Fields Medal “for his contributions to bridging probability, representation theory and algebraic geometry.”

Figure 4. Andrei Okounkov and a Young diagram.